

My on-going research project

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My on-going research project (with respect to the theme of the workshop) includes :

(i) the stability of several political organizations : this is the topic I wish to present at the workshop

(ii) The problem of rational behaviour in a voting context and

(iii) The measure of voters' power in committees.

JEL classification numbers : C62, C70, C71, D71, D72

Keywords : Democracy, Voting game; core; Social decision system; Voting game with abstention, Political stability, Power index, Prudent standard of behaviour,

1 The Stability of political organizations described by voting games with abstention

1.1 The main problem of social choice theory

The major problem in social choice theory is to define a "democratic" voting system. It is well known that common voting procedures have surprising and paradoxical properties. We shall show in the simple example below how the voting procedure can deeply affect the outcome of an election.

Consider a committee of 27 members (voters) choosing one out of four candidates in competition x , y , z and t . Assume that the preferences (individual rankings) are described in the following table.

for 5 voters	$x > y > z > t$,	for 2 voters	$t > y > x > z$,	for 8 voters	$z > y > x > t$
for 4 voters	$x > z > y > t$,	for 6 voters	$t > y > z > x$,	for 2 voters	$t > z > y > x$

This means, for example that according to 5 voters, x is the best candidate, y is the second, z the third and t is the worst. According to 4 voters, x is the best, z the second, y the third and t the worst, and so on...

Let us find the outcome with respect to some classical social choice procedures in the literature.

1.1.1 Procedure 1 : Plurality voting

According to the plurality voting, the outcome is the candidate ranked first by the largest number of voters. x is ranked first by 9 voters, y is ranked first by no voter, z by 8 voters and finally t , by 10 voters. Thus with respect to the plurality voting, t is elected. Furthermore, the social ranking is $t > x > z > y$.

Remark : Among the 27 voters, 17 prefer x to t !

1.1.2 Procedure 2 : two round majority voting

According to this procedure, first, we select the two best placed alternatives and apply majority voting within a second vote. In our example, t will be opposed to x at the second round. Since 17 voters prefer x to t , the elected candidate is x .

Note : Again, 18 voters prefer y to x !

1.1.3 Procedure 3 : Borda's procedure

- Each of the m position (m is the number of candidates, which is 4 in our example) is graded : m for the first, $m - 1$ for the second, ..., and 1 for the last. We look at the preference ordering of each voter, each candidate is graded accordingly. We add the grades, each candidate then receives a number of points. The candidate with the largest number of points wins.

In our example, x receives $(9 \times 4) + (10 \times 2) + (8 \times 1) = 64$ points, y receives $(21 \times 3) + (6 \times 2) = 75$. As well, z , 74 points and t , 57 points. Thus, the winner is y and furthermore, the social ranking is $y > z > x > t$

Remark : More than the half of voters prefer z to y .

1.1.4 Procedure 4 : Condorcet rule (majority rule)

A candidate is Condorcet winner if, confronted to any other candidate, it comes before in more than half of the orderings. Let us denote by m_{ab} , the number of voters who rank a before b . In our example, we have :

$$m_{xt} = 17, m_{yx} = 18, m_{yt} = 17, m_{zx} = 16, m_{zy} = 14 \text{ and } m_{zt} = 17$$

Thus the winner is z (z defeats x , y and t). Furthermore, the social ranking is $z > y > x > t$.

The main problem of this procedure is that it is indecisive. Indeed the Condorcet winner may not exist as we can see in the following example in which we modify the profile.

10 voters $x > y > z > t$ 9 voters $y > z > t > x$ 8 voters $z > t > x > y$

The (first) example above show how the procedure may affect the outcome of an election. Each candidate may be elected . This yields the problem of criteria to be satisfied by an "acceptable" voting procedure.

Indeed Arrow (1951) showed that there exists no voting rule satisfying some minimal conditions.

1. **Pareto criteria** : If all voters prefer x to y , then x should socially be preferred to y (y cannot be elected)
2. **Independence criteria** (independence of irrelevant alternatives) : The social ranking on two candidates x and y depends solely on individual rankings on those candidates.
3. **Non dictatorial criteria** : The social ranking should not always be dictated by a fixed voter (called dictator)

A *social welfare function* assigns to every preference profile a social ranking. Arrow's impossibility theorem is stated as follows.

Theorem 1 : There exists no social welfare function satisfying the Pareto, Independence and non dictatorial criteria.

The Independence condition is the most commented property in the litterature. It has been shown (Andjiga and Moulen, 1988, 2003) that any social welfare function satisfying the independence condition is induced by a distribution of power (constitution) to different coalitions in the society.

A constitution Δ on N assigns to any couple of distincts candidates (x, y) a family $\Delta(x, y)$ of couples of disjoints coalitions with the following interpretation. If y is under consideration and x is a candidate to the replacement of y , if all the members of S vote for the replacement and those of T vote against, then the replacement will be enforced no matter what the other voters (members of the complement of $(S \cup T)$ do).

The social welfare function induced by a given Δ on N is defined below.

Let R be a profile of preferences, x and y two candidates.

We denote by $N_{xy}^+(R)$ the set of voters who strictly prefer x to y (Thus, $N_{yx}^+(R)$ is the set of voters who strictly prefer y to x)

1. x dominates y if $(N_{xy}^+(R), N_{yx}^+(R))$ belongs to $\Delta(x, y)$.
2. x dominates y means that x is socially ranked before (or preferred to) y .

Theorem 2 : A social welfare function satisfies the Independence criteria if and only if it is induced by a constitution on N .

If a constitution satisfies some conditions (neutrality, paretian, monotonicity), it can simply be described by a given set of coalitions \mathcal{V} called winning coalitions or majorities with the following interpretation. $S \in \mathcal{V}$ means that if a given y is under consideration and x is a candidate for the replacement of y , if all the members of S vote for the replacement and then the replacement will be enforced no matter what the other voters (members of the complement S do). In this case, the triple (N, A, \mathcal{V}) is called a voting game

1.2 The voting games with abstention case

N refers to the *electoral body* or *the society*, A is the set of *candidates*. The set of subsets of N is denoted by $\mathcal{P}(N)$ while the set of non-empty subsets of N is 2^N . $|S|$ stands for the number of elements of the finite set S . S^c denotes the set of voters who are not members of S that is, $S^c = N \setminus S$. Let us first recall the notion of voting game.

Definition 1. 1) A simple or voting game is any $G = (N, A, \mathcal{V})$ where $\emptyset \neq \mathcal{V} \subset 2^N$ satisfies the following monotonicity condition : for all $S, T \in 2^N$, if $S \in \mathcal{V}$ and $S \subset T$, then $T \in \mathcal{V}$. Any $S \in \mathcal{V}$ is called a majority or a winning coalition

2) A voting game G is said to be proper if : for all $S \in 2^N$, if S is winning then S^c is losing.

In a simple game, a majority enjoys the following power : if each member of S votes for a bill, then the bill pass no matter what the members of $N \setminus S$ do. For example, in the case of the strict majority rule, a coalition is winning if and only if $|S| > \frac{|N|}{2}$.

The monotonicity condition says that a coalition containing a winning coalition is also winning.

Many institutions are modelled as simple games. These are voting institutions where voters vote by "Yes" or "No" on a single issue. Examples are EU Council of Ministers, US Federal System, Canadian Constitution Parliaments, Governing boards ...

The relative majority rule is a well known classical voting context which cannot be described by a simple game. In this context, if S_1 is the set of members who vote for a bill, S_2 is the set of voters who abstain and S_3 the set of those who vote against. Hence, the bill is enforced if and only if $|S_1| > |S_3|$. The sequel (S_1, S_2, S_3) is called a tri-partition of the society. We denote by \mathcal{N} the set of all tripartitions of N .

Rubinstein's model of voting game with abstention or social decision system is defined as follows :

Definition 2 : A Social decision system (Sds) is a triple $\mathcal{H} = (N, A, \mathcal{W})$ where \mathcal{W} is a subset of \mathcal{N} which satisfies :

(i) : $\mathcal{W} \neq \emptyset$,

(ii) For all tri-partitions $(S_1, S_2, S_3), (T_1, T_2, T_3)$, if $(S_1, S_2, S_3) \in \mathcal{W}$, $S_1 \subset T_1$ and $S_1 \cup S_2 \subset T_1 \cup T_2$ then $(T_1, T_2, T_3) \in \mathcal{W}$.

(iii) for all tripartition (S_1, S_2, S_3) , if (S_1, S_2, S_3) is winning, then $(S_3, \emptyset) \notin \mathcal{W}$.

Each element (S_1, S_2, S_3) of \mathcal{W} is called a winning tripartition.

For the interpretation, assume that b is the status quo and that a is a candidate for the replacement of b . (S_1, S_2, S_3) is winning means that if all the voters in S_1 vote for the replacement, all those of S_2 abstain and all members of S_3 vote against the replacement, then the replacement will be enforced.

he following result shows that the class of voting games with abstension includes the class of voting game.

Proposition 3 : Any voting game $G = (N, A, \mathcal{V})$ can be associated with a voting game with abstention

Now let us define the core of a voting game with abstension.

We recall that the preference of each voter i is expressed as a weak order, (that is a rational binary relation) \succsim_{R^i} on the set of candidates A . \succ_{R^i} denotes the strict component of \succsim_{R^i} . The list $(R^i)_{i \in N}$ also denoted by R is called a *profile*.

Definition 3 : Let $G = (N, A, \mathcal{V})$ be a simple game, $(y, x) \in \mathcal{X}$, R a profile and $S \in 2^N$.

1. x dominates y in (G, R) denoted by $x \succ y$ if there exists $S \in \mathcal{V}$ such that $\forall i \in S, x \succ_{R^i} y$.
 2. A candidate y is said to be stable if there does not exist another candidate x such that x dominates y .
 3. The core of the vote denoted by $\mathcal{C}(G, R)$ is the set of stable candidates in (G, R) .
 4. A Voting game is said to be stable if and only if for every profile R^N , the core of (G, R) is non empty.
- Assume that for a given profile, the core is empty. Thus no candidate is stable. In other word, if a candidate x_1 is under consideration, he will be replaced with another candidate say x_2 , x_2 in turn will be replaced with x_3 , and so on until we reach another candidate x_q that is dominated by x_1 . This is the reason of the instability of the constitution.

The above definition of the core was extended to Sds by Rubinstein (1980). Before that, we need the following notation :

Let S be a coalition, x and y be two candidates, and R a profile. We denote by $S_{xy}^\sim(R)$ the set of those who are indifferent between x and y .

Definition 4 : Let $\mathcal{H} = (N, A, \mathcal{W})$ be an Sds, (y, x) a couple of distincts candidates, R a profile.

1. x dominates y in (\mathcal{H}, R) denoted by or $x \succ y$ if $(N_{xy}^+(R), N_{xy}^\sim(R^N), N_{yx}^+(R)) \in \mathcal{W}$.
2. y is said to be stable in (\mathcal{H}, R) if there does not exist a candidate x such that x dominates y .
3. The core of the VGA (\mathcal{H}, R) denoted by $\mathcal{C}(\mathcal{H}, R)$ is the set of stable or non dominated candidates in (\mathcal{H}, R) .

4. A VGA is said to be stable if and only if for every profile R , the core of (\mathcal{H}, R) is non empty.

The main problem of the core is that it may be empty. Indeed, consider the simple case below.

There are three players 1, 2, 3 and three candidates x , y and z . We consider the strict majority rule. That is a coalition is winning if and only if it contains two voters (out of three). This means that the minimal winning coalitions are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$.

Consider the following profile in which voter 1 ranks x first, y second and z third, voter 2 ranks y first, z second and x third, voter 3 ranks z first, x second and y third. We can found that no candidate is stable.

Indeed x will be replaced by z via coalition $\{2, 3\}$, y will be replaced by x via coalition $\{1, 3\}$, z will be replaced by y via the coalition $\{1, 2\}$. Thus the core is empty. This phenomenon is known as the paradox of voting.

The purpose of this paper is to find a necessary and sufficient condition under which the core of a VGA is always non empty for every profile of preferences. This problem was elegantly solved by Nakamura (1979) in the case of voting game as follows.

Given a simple game $G = (N, A, \mathcal{V})$, let $\Gamma(\mathcal{V})$ be the set of family of coalitions which intersection is empty.

The Nakamura's number of G denoted by $\nu(G)$ is defined as the smallest number of coalitions which intersection is empty.

The Nakamura's theorem is stated as follows :

Theorem 4 : Let $G = (N, A, \mathcal{V})$ be a voting game. Then, G is stable if and only if $\nu(G) > |A|$

Nakamura's theorem provides an algebraic characterization of the set of political rules or constitutions under which the society would allow a given number of candidates to compete without running the risk of being thrust in an endless cycle of political changes. For a given constitution, this result also provides an upper bound for the number of candidates that the society can allow to compete while at the same time avoiding political instability.

We extend Nakamura's theorem to voting game with abstension.

$$\Omega(\mathcal{W}) = \{ \mathcal{W}' = \{(S_1^1, S_2^1, S_3^1), \dots, (S_1^k, S_2^k, S_3^k)\} \subseteq \mathcal{W} : \forall p \in \{1, 2, \dots, k\}, S_1^p \cap (\bigcap_{\tau \neq p} S^\tau) = \emptyset \}$$

where $S^\tau = S_1^\tau \cup S_2^\tau$

We define the Nakamura's number of \mathcal{H} denoted by $\mu(\mathcal{H})$ as follows :

$$\mu(\mathcal{H}) = \begin{cases} +\infty & \text{if } \Omega(\mathcal{W}) = \emptyset \\ \min\{|\mathcal{A}| / \mathcal{A} \in \Omega(\mathcal{W})\} & \text{if } \Omega(\mathcal{W}) \neq \emptyset \end{cases} .$$

We therefore state the Nakamura's theorem for an Sds as follows :

Theorem 5 : Let $\mathcal{H} = (N, A, \mathcal{W})$ be an Sds. Then, \mathcal{H} is stable if and only if $\mu(\mathcal{H}) > |A|$.

We prove that our theorem is a generalization of Nakamura's.

1.3 Some applications

We determine the Nakamura number of the relative majority voting rule, the vote at the United States Senate in the case of passing an ordinary bill and the vote at the United Nations Security Council

1.3.1 The relative majority voting

According to the relative majority voting rule, a tripartition $\mathcal{S} = (S_1, S_2, S_3)$ is winning if and only if $|S_1| > |S_3|$. That is, the a decision is adopted if and only if the number of voters who favor the decision is greater than the number of those who are against it.

Theorem 6 : A relative majority game is stable if and only if there are :

- 1) two voters; or
- 2) two candidates and any number of voters different from four; or
- 3) three candidates and four voters.

Thanks to the result above, as soon as the number of candidates is at least four, if the voting mechanism in a society is the relative majority voting then it is easy for that society to enter in an endless cycle of political instability.

1.3.2 The vote at the United States Senate in the case of passing an ordinary bill

The United States Senate (USS) counts 100 senators and the vice-president who leads the Senate. Voting rules vary depending on the nature of the decision to be made. In the case of passing an ordinary bill or an amendment, a decision is made when the number of senators who cast a "yes" vote is strictly greater than the number of senators who cast a "no" vote, and vice-versa; in case of equality, the vice-president of the United States (vp for short) casts a tie-breaking vote. This social choice institution is a voting game with abstention modelled as follows :

Let $\mathcal{S} = (S_1, S_2, S_3)$ be an ordered 3-partition of the USS :

$$\mathcal{S} \text{ is winning if and only if } |S_1| > |S_3 \setminus \{vp\}|.$$

The following result states that the vote of the United States Senate in the case of passing an ordinary bill or an amendment is stable if and only if the number of candidates in competition is smaller than three.

Theorem 7 : The vote of the United States Senate in the case of passing an ordinary bill or an amendment is stable if and only if the number of candidates is smaller than 3.

1.3.3 The vote at the United Nations Security Council

The United Nations Security Council (UNSC) is made up of 10 non-permanent members and 5 permanent members. A resolution is enforced if at least 9 members support it and no permanent member is explicitly opposed to it. Let us denote by P the set of all permanent members and \bar{P} the set of non-permanent members. The vote of the UNSC, is modelled as follows.

Let $\mathcal{S} = (S_1, S_2, S_3)$ be an ordered 3-partition :

$$\mathcal{S} \text{ is winning if and only if } |S_1| \geq 9 \text{ and } S_3 \cap P = \emptyset$$

Theorem 8 : The vote of the United Nations Security Council is stable if and only if it involves at most 9 candidates.

We can assert that the vote at the UNSC is always stable. Indeed, this institution would allow up to nine candidates to compete without running the risk of political instability.

2 Rational behaviour in a voting context

We consider social choice framework modeled by a voting game, which is a list (N, A, \mathcal{V}) . Voters are invited to deliberately choose a candidate as the social choice. The vote takes place within an iterative framework, meaning that if a winning coalition replaces the status quo a by a contestant b , b becomes the new status quo, and the vote begins afresh, until a candidate is reached that no winning coalition is willing to replace. This means that the elimination of the status quo can be followed by a succession or a chain of deviations. In such an institutional framework, common sense implies that voters should choose rationally in order to maximize their utility or welfare. More clearly, this means that before a voter joins a coalition to eliminate the status quo a and enforce a new candidate b , he should make sure that any potential succession of deviations from the new status quo b will not lead to a final elected candidate which is worse than the initial status quo a from her point of view. Such a voter can be thought as exercising "prudence". Modeling *prudent standards of behavior* in coalitional voting game is the problem we propose to solve in our study.

To address the possibility of non-emptiness of the core, Rubinstein (1980) first noted the lack of farsightedness in the core, and defined the so called dominance with precaution of order 1. This later is prudent, and moreover the set of undominated candidates, called the stability set is never empty when individuals preferences are strict orders.

We prove that these positive results do no longer hold (neither the stability set is prudent nor it is always nonempty) in the general case where individuals preference are weak orders. Next, we define a new standard of behaviour, the so called P-dominance and we study its properties.

3 The measure of voters' power in committees.

In any political organization in which outputs are produced by groups of individuals, the question of how much performant or empowered is each individual in the organization has gained increasing interest over the past sixty years. Two approaches to this question have been developed in the literature. The quantitative approach, which consists of a real-valued function $f : N \rightarrow R$ over the set of players N , assigns to each player a numerical value measuring his power. This approach dates back to Penrose (1946) pioneering work on voting power and has been widely adopted by most of the subsequent researches in the field (Shapley and Shubik 1954; Banzhaf 1965, see Andjiga, Chantreuil and Lepelley (2003) for a complete review of the literature on numerical power theories). The qualitative approach introduced by Isbell (1958) consists of a binary relation called "desirability relation" or "influence relation", which ranks voters according to their influence in a vote. The common ground to all these theories is that they have been essentially studied in the basic framework of simple games where players can only cast a "yes" or a "no" vote.

However in most real-life cooperative organizations, several levels of participation are generally observed. In voting games for instance, "abstention" as an intermediate option between a "yes" and a "no" vote is widely observed in most elections. This fact prompted scholars to think of a more general model of voting games, and to accordingly re-examine some of the power theories studied in the former model. Rubinstein (1980) introduced the more general model of "social decision systems, Freixas and Zwicker (2003) further generalized the model of simple games to games where voters have j possible levels of approval in the input, thus partitioning the society N into j coalitions, each associated with a certain level of approval, and each possible partition facing k possible levels of approval in the output. Both j and k are naturally ordered, defining the so-called (j, k) voting games. In this more general and important class of games, voting games constitute the class of $(2, 2)$ voting games and SDSs constitute the class of $(3, 2)$ voting games. Felsenthal and Machover (1997) generalized the Shapley-Shubik and Banzhaf power indices to $(3, 2)$ simple games. Freixas (2005a, 2005b) further generalized them to (j, k) simple games. In our study, we attempt to :

- 1) generalize the influence relation to (j, k) voting games and study its properties
- 2) analyse several power measures available in the literature as real tools to influence the voting outcome of an election.